Grading guide, Pricing Financial Assets, June 2017

1. Consider an economy with securities available for investment at time $t = 0$ and with ultimate payment at time $t = T$. A stock and a zero coupon bond can be purchased at $t = 0$ at prices S_0 and e^{-rT}, respectively, where r is a constant. Neither of these securities has payments between time 0 and T.

   Assume that at time T the realization of two possible states of the world is revealed. The stock has a payoff of uS_0 and dS_0, depending on the state, where $u > \mathrm{e}^{rT} > d \geq 0$ are constants. The bond has a payment of 1 at $t = T$, irrespective of the state.

   (a) Explain what is meant by risk neutral probabilities and (Arrow-Debreu) state prices and determine these for this economy.

   (b) What is the forward price at $t = 0$ for delivery of the stock at $t = T$? What is the arbitrage argument?

   (c) Consider a forward contract on a stock maturing at time T. Compare the pricing of a European call option on the stock with the price of a European call on the forward contract on the stock, all maturing at T and having the same strike K.

Solution:

(a) State prices ψ are implicit prices that satisfy that current prices of each security is the sum of the time and state dependent payoffs of the security and corresponding (security independent) state prices. Risk neutral probabilities q are probabilities that satisfy that current prices of each security is the discounted expected payoff using of these (security independent) probabilities and discounting with a risk free rate. Existence of both follows if there are no arbitrage opportunities in the economy.

Let for ease of notation $a = \mathrm{e}^{rT}$. In the concrete case the state prices (ψ_u, ψ_d) satisfies the two equations

$$1 = a\psi_u + a\psi_d \tag{1}$$
$$1 = u\psi_u + d\psi_d \tag{2}$$

with the solution $\psi = (\psi_u, \psi_d) = (a^{-1}\frac{a-d}{u-d}, a^{-1}\frac{u-a}{u-d})$ which are well defined and positive with our assumptions. The risk neutral probabilities are $q = (q_u, q_d) = a\psi$, and note that $q_u + q_d = 1$.

(b) Consider two strategies to end with the stock at time $t = T$: A) Enter into a forward contract. There is no cash flow at time $t = 0$. You pay the agreed forward price at $t = T$. B) Borrow S_0 at the risk free rate, buy the stock and hold it to $t = T$, where you repay the loan by paying aS_0. Here also there is no net cash flow at time $t = 0$. In the absence of arbitrage opportunities the forward price contracted at $t = 0$ is thus $F_0^T = aS_0$.

(c) Since the forward contract will expire with $F_T^T = S_T$ the two call options will have the same price.

2. Let $V(t)$ denote the probability of a borrower not defaulting before time $t \geq 0$, $V(0) = 1$.

   (a) We will often assume that $V(t)$ is weakly decreasing in t. Why?

   (b) Assume that V is differentiable. Define and interpret the continuously compounded hazard rate, $\lambda(t)$.

   (c) What is a ratings transitions matrix? What is the typical structure af such a matrix?

(d) What assumptions are needed to use a ratings transition matrix to derive an estimate of the likelihood of a default of a rated issuer within a certain time horizon? And will this estimate be a real world or a risk neutral probability?

Solution:

(a) The assumption that $V(t)$ is weakly decreasing reflect that we model defaults that accumulate over time - we do not include a modelling of a borrower getting out of default.

(b) The conditional default probability from t to $t + \Delta t$ is

$$\frac{V(t) - V(t + \Delta t)}{V(t)}$$

Implicitly defining the hazard rate

$$-\lambda(t)V(t)\Delta t = V(t + \Delta t) - V(t)$$

we get by taking limits as Δt goes to zero

$$\frac{\partial V(t)}{\partial t} = -\lambda(t)V(t)$$

or

$$V(t) = \mathsf{e}^{-\int_0^t \lambda(\tau)d\tau}$$

Thus the hazard rate measures the rate at which non-default or survival degrades over time.

(c) A rating transitions matrix is a matrix that contains historical frequencies or modelled probabilities describing the likelihood of an obligor of a particular rating having different ratings after a specified length of time. If augmented with a state describing default and disregarding the possibility that the obligor ceases to be rated, the matrix is square with numbers between 0 and 1. The default state is absorbing if we assume that obligors stay in default once defaulted. In practice the matrix has large weights on the diagonal (assume the time period to be a year or shorter). Also the likelihood of staying with a particular rating is typically larger the better the rating category.

(d) Using an empirical rating transition matrix requires that we deal with obligors that cease to be rated. One strategy is to assume that this likelihood is independent of the rating category (which is doubtful). Here you just remove the non-rated category and increase the other frequencies proportionally. For modelling future defaults we then may assume (also doubtfully) that the rating transition matrix is constant. In any case our estimate will be made under the empirical, physical measure $\mathbb{P}$.

3. Let $R(t, T)$ denote the continuously compounded yield at time t of a zero coupon bond that matures at time $T > t$. Assume that R follows an Ito-process of the form

$$dR = \mu dt + \sigma dz$$

where z is a Brownian motion, and μ and $\sigma > 0$ are bounded functions of R and t.

(a) For a bond with given maturity T show that the volatility of the bond price $P(t, T)$ will converge to zero as t converges to T.

(b) Now assume $\mu = m(k - R)$ and $\sigma = sR$, where m, k and s are positive constants. What is the process followed by the bond price?

Solution:

(a) Let $P(t,T) = \mathsf{e}^{-(T-t)R}$. Using Ito's lemma we have that the volatility of the price is

$$\frac{\partial P}{\partial R}\sigma = -(T-t)P(t,T)\sigma$$

Since σ stays bounded, and $P(t,T)$ converges to 1 when t converges to T, we have the required result.

(b) We find the partials

$$\frac{\partial P}{\partial t} = RP \tag{3}$$
$$\frac{\partial P}{\partial R} = -(T-t)P \tag{4}$$
$$\frac{\partial^2 P}{\partial R^2} = (T-t)^2 P \tag{5}$$

and get by Ito's lemma the process

$$dP = (R - (T-t)m(k-R) + 0.5s^2R^2(T-t)^2)Pdt + (T-t)sRPdz$$

By the symmetry of the Brownian increment the sign on the last part is immaterial.